

# Gravitational collapse of spherically symmetric perfect fluid with kinematic self-similarity

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# Abstract

Analytic spherically symmetric solutions of the Einstein field equations coupled with a perfect fluid and with self-similarities of the zeroth, first and second kinds, found recently by Benoit and Coley [Class. Quantum Grav. **15**, 2397 (1998)], are studied, and found that some of them represent gravitational collapse. When the solutions have self-similarity of the first (homothetic) kind, some of the solutions may represent critical collapse but in the sense that now the “critical” solution separates the collapse that forms black holes from the collapse that forms naked singularities. The formation of such black holes *always* starts with a mass gap, although the “critical” solution has homothetic self-similarity. The solutions with self-similarity of the zeroth and second kinds seem irrelevant to critical collapse. Yet, it is also found that the de Sitter solution is a particular case of the solutions with self-similarity of the zeroth kind, and that the Schwarzschild solution is a particular case of the solutions with self-similarity of the second kind with the index  $\alpha = 3/2$ .

## I. INTRODUCTION

Gravitational collapse of a realistic body has been one of the most important and thorny subjects in General Relativity (GR) since the early times of GR [1]. Quite recently, thanks to Choptuik's numerical discovery of critical phenomena in the threshold of black hole formation [2], the subject has attracted further attention. As a matter of fact, it is so attractive that *Critical Phenomena in Gravitational Collapse* has been already a very established sub-area in GR [3,4]. From all the work done so far, the following seems clear: (a) The critical solution and the two dimensionless constants  $\Delta$  and  $\gamma$  are universal only with respect to the same matter field, and usually are matter-dependent. (b) The universality of the critical solution and the exponent  $\gamma$  now are well understood in terms of perturbations [5], while the physical origin of  $\Delta$  still remains somewhat of a mystery. The former is closely related to the fact that the critical solution has only one unstable mode. This property now is considered as the main criterion for a solution to be critical. (c) The critical solutions can have discrete self-similarity (DSS) or homothetic self-similarity (HSS) <sup>1</sup>, or none of them, depending on the matter fields and regions of the initial data spaces. So far, in all the cases where the critical solution either has DSS or HSS, the formation of black holes *always* turns on with zero mass, the so-called Type II collapse, while in the cases in which the critical solution has neither DSS nor HSS, the formation *always* turns on with a mass gap, the so-called Type I collapse [3,4].

In the Type II collapse, it is the usual belief that the fact that black hole starts to form with an infinitesimal mass is closely related to the fact that the problem concerned is of scale invariance, for example, the Einstein equations coupled with a massless scalar field. When the scalar field is massive, the corresponding field equations are scale-invariant only

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<sup>1</sup>In the literature, homothetic self-similarity has been also called continuous self-similarity. However, in order to distinguish it from the self-similarity of the other kinds, in this paper we shall refer it as homothetic self-similarity, or self-similarity of the first kind.

asymptotically [3]. For a perfect fluid with the equation of state  $p = k\rho$ , the corresponding Einstein field equations are also of scale invariance. As a result, in all these cases critical phenomena of Type II collapse were found, and in the case of the scalar field the critical solution has DSS, while in the case of perfect fluid, the critical solutions have HSS [2,6,7].

It is known that homothetic self-similarity is a particular case of kinematic self-similarity [8]. In fact, the latter consists of three kinds, the zeroth kind, the first (homothetic) kind, and the second kind. Thus, a natural question is: Can critical solutions have self-similarity of the other kinds?

In this paper, we shall study this problem for the gravitational collapse of perfect fluid with kinematic self-similarity. As a matter of fact, several classes of such analytic solutions to the Einstein field equations are already known [9]. So, here we shall study these solutions in some details and pay particular attention on critical solutions. Finding critical solutions usually consists of two steps, one is first to find a generic family (or families) of solutions, characterized, say, by a parameter  $p$ , such that when  $p > p^*$  the collapse forms black holes, and when  $p < p^*$  it does not. Once such solutions are found, one needs to make perturbations of the solution  $p = p^*$  and to study the spectrum of their modes. If the solution has only one unstable mode, then by definition this solution is a critical solution, and the exponent  $\gamma$  is given by

$$\gamma = \frac{1}{|\sigma_1|}, \quad (1.1)$$

where  $\sigma_1$  is the unstable mode [5]. In this paper, we shall consider only the first part of the problem, and leave the study of perturbations to another occasion. Specifically, the paper is organized as follows: In Sec. II we shall give a brief introduction to kinematic self-similarity, and in Sec. III we shall study the Benoit-Coley (BC) solutions with self-similarity of the zeroth kind, while in Sec. IV the BC solutions with self-similarity of the first and second kinds will be studied. The paper is closed with Sec. V, in which our main conclusions are presented. An appendix is also included, where the Einstein field equations are written in terms of self-similar variables.

## II. SPHERICALLY SYMMETRIC SPACETIMES WITH KINEMATIC SELF-SIMILARITY

Self-similarity refers to the fact that the spatial distribution of the characteristics of motion remains similar to itself at all times in which all dimensional constant parameters entering the initial and boundary conditions vanish or become infinite [10]. Such solutions describe the “intermediate asymptotic” behavior of solutions in the region where a solution no longer depends on the details of the initial and/or boundary conditions.

Cases in which the form of the self-similar asymptotes can be obtained from dimensional considerations are referred to as self-similarity of the *first (homothetic) kind* [10]. Solutions of the first kind were first studied by Cahill and Taub in GR for a perfect fluid [11]. They showed that the existence of self-similarity (of the first kind) could be formulated invariantly in terms of a homothetic Killing vector,  $\xi^\mu$ , which satisfies the conformal Killing equation,

$$\mathcal{L}_\xi g_{\mu\nu} = 2g_{\mu\nu}, \quad (2.1)$$

where  $\mathcal{L}$  denotes Lie differentiation along  $\xi^\mu$ . From the above it can be shown that

$$\mathcal{L}_\xi G_{\mu\nu} = 0. \quad (2.2)$$

For a perfect fluid with the energy-momentum tensor (EMT) given by

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu}, \quad (2.3)$$

it can be shown that it is consistent with Eq.(2.2) if we require

$$\mathcal{L}_\xi u^\mu = -u^\mu, \quad \mathcal{L}_\xi \rho = -2\rho, \quad \mathcal{L}_\xi p = -2p. \quad (2.4)$$

Hence, in this case “geometric” self-similarity and “physical” similarity coincide, although this does not need to be so in more general cases [8,12]. Applying the above to the spacetimes with spherical symmetry,

$$ds^2 = r_1^2 \left\{ e^{2\Phi(t,r)} dt^2 - e^{2\Psi(t,r)} dr^2 - r^2 S^2(t,r) d\Omega^2 \right\}, \quad (2.5)$$

where  $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ , Cahill and Taub found that the condition (2.1) requires

$$\Phi(t, r) = \Phi(\xi), \quad \Psi(t, r) = \Psi(\xi), \quad S(t, r) = S(\xi), \quad (2.6)$$

where

$$\xi = \frac{r}{-t}. \quad (2.7)$$

The corresponding homothetic Killing vector  $\xi^\mu$  is given by

$$\xi^\mu \frac{\partial}{\partial x^\mu} = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \quad (2.8)$$

Note that in writing the metric (2.5) we had multiplied a factor  $r_1^2$  to the usual spherical metric, so that the metric coefficients  $\Phi$ ,  $\Psi$ ,  $S$  and the coordinates  $t$ ,  $r$ ,  $\theta$  and  $\varphi$  now are all *dimensionless*, where we assume that  $r_1$  has the dimension of length. It is found that this choice will simplify the dimensional analysis to be given below. The corresponding Einstein tensor and Einstein field equations are given in terms of both  $t$ ,  $r$  and  $x$ ,  $\tau$  in the Appendix, where  $x$  and  $\tau$  are the self-similar variables that are functions of  $t$  and  $r$ . Their explicit definitions in each case are given in the Appendix.

The existence of self-similarity of the first kind is closely related to the conservation laws and to the invariance of the problem with respect to the group of similarity transformations of quantities with independent dimensions, in which case a certain regularity of the limiting process in passing from original non-self-similar regime to the self-similar regime is assumed implicitly. However, in general such a passage does not need to be regular. Consequently, the expressions for the self-similar variables are not determined from dimensional analysis. Such solutions are then called self-similar solutions of the *second kind*. A characteristic of these solutions is that they *contain dimensional constants* that are not determined from the conservation laws [10]. Using these arguments to a perfect fluid (2.3), Carter and Henriksen [8] gave the notion of *kinematic self-similarity* with its properties,

$$\mathcal{L}_\xi h_{\mu\nu} = 2h_{\mu\nu}, \quad \mathcal{L}_\xi u^\mu = -\alpha u^\mu, \quad (2.9)$$

where  $h_{\mu\nu}$  is the project operator, defined by

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu, \quad (2.10)$$

and  $\alpha$  is an arbitrary *dimensionless* constant. When  $\alpha = 1$ , it can be shown that the kinematic self-similarity reduces to the self-similarity of the first kind (homothetic self-similarity). When  $\alpha \neq 1$ , Carter and Henriksen argued that this would be a natural relativistic counterpart of self-similarity of the second kind ( $\alpha \neq 1$ ), and of the zeroth kind ( $\alpha = 0$ ), in Newtonian Mechanics.

Applying the above to the spherical case, Carter and Henriksen found that the metric coefficients  $\Phi$ ,  $\Psi$  and  $S$  should also take the form of Eq.(2.6) but with the self-similar variable  $\xi$  and conformal vector  $\xi^\mu$  now being given, respectively, by

$$\xi^\mu \frac{\partial}{\partial x^\mu} = \alpha t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}, \quad \xi = \frac{r}{(-t)^{1/\alpha}}, \quad (\alpha \neq 0), \quad (2.11)$$

for the second kind, and

$$\xi^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}, \quad \xi = r e^{-t}, \quad (\alpha = 0), \quad (2.12)$$

for the zeroth kind. Comparing Eq.(2.8) with Eq.(2.11) we find that the self-similarity of the first kind can be considered as a particular case of the one of the second kind. In this paper we shall do so, although, as we mentioned above, the physics is quite different in the two cases. In particular, when the coordinates  $t$  and  $r$  are rescaled,  $t' = ct$  and  $r' = cr$ , where  $c$  is a constant,  $\xi$  is unchanged only for the homothetic case  $\alpha = 1$ .

### III. ANALYTIC SOLUTIONS OF PERFECT FLUID WITH SELF-SIMILARITY OF THE ZEROth KIND AND THEIR PHYSICAL INTERPRETATIONS

The solutions to be studied in this section are those given by Eqs.(2.54) - (2.57) in [9]. Note that the expression for the function  $S(x)$  given there is not correct. As a matter of fact, setting  $\Phi = 0$ , we find that Eq.(A.27) yields

$$e^{2\Psi} = (1 + y)^2 S^2, \quad (3.1)$$

while Eq.(A.22) is satisfied automatically. Substituting Eq.(3.1) into Eq.(A.26), we obtain

$$y_{,xx} + 3yy_{,x} = 0, \quad (3.2)$$

which has the first integral

$$2y_{,x} + 3y^2 + p_0 = 0, \quad (3.3)$$

where  $x = \ln(\xi)$ , and  $p_0$  is a *dimensionless* constant, in contrast to the claim given in [9].

Then, it can be shown that the corresponding perfect fluid is given by <sup>2</sup>

$$\rho = \frac{y(3y - p_0)}{r_1^2(1 + y)}, \quad p = \frac{p_0}{r_1^2}, \quad u_\mu = r_1 \delta_\mu^t, \quad (3.4)$$

where  $y(x) \equiv S_{,x}/S$ . Depending on the sign of  $p_0$ , Eq.(3.3) has physically different solutions.

In the following let us consider them separately.

#### A. Case $p_0 = 0$

In this case, it can be shown that Eq.(3.3) has the solution,

$$y(x) = \frac{2}{3(x + x_0)}, \quad S(x) = S_0(x + x_0)^{2/3}, \quad (3.5)$$

where  $S_0$  and  $x_0$  are integration constants. Without loss of generality, we can set  $S_0$  equal to one by a conformal transformation. In the following we shall assume that this is always done whenever it is applicable. Substituting the above expressions into Eq. (3.4), we find that

$$\rho = \frac{4}{r_1^2(x + x_0)[3(x + x_0) + 2]}, \quad p = 0, \quad (3.6)$$

which shows that in this case the solutions represent a dust fluid. Thus, these solutions must belong to the general Tolmann-Bondi class [13].

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<sup>2</sup>In this paper the units will be chosen such that the Einstein coupling constant  $\kappa [\equiv 8\pi G/c^4] = 1$ .



# FIGURES

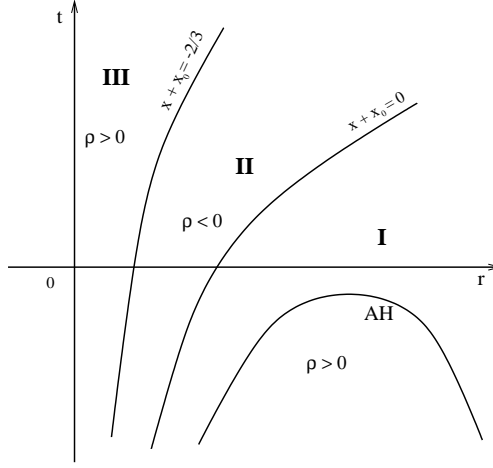


FIG. 1. The spacetime described by the solutions of Eq.(3.5) in the text in the  $(t, r)$ -plane. It is singular on the hypersurfaces  $x + x_0 = 0$  and  $x + x_0 = -2/3$ , which divide the whole spacetime into three disconnected regions:  $I = \{x^\mu : x + x_0 \geq 0\}$ ,  $II = \{x^\mu : -2/3 \leq x + x_0 \leq 0\}$ , and  $III = \{x^\mu : x + x_0 \leq -2/3\}$ .

From Eq.(3.6) we can see that the spacetime is singular on the hypersurfaces  $x + x_0 = 0$  and  $x + x_0 = -2/3$ . These two hypersurfaces divide the whole spacetime into three regions, I, II, and III, where  $I = \{x^\mu : x + x_0 \geq 0\}$ ,  $II = \{x^\mu : -2/3 \leq x + x_0 \leq 0\}$ , and  $III = \{x^\mu : x + x_0 \leq -2/3\}$  [See Fig. 1]. In region II, the energy density of the fluid  $\rho$  is negative, and the physics of the spacetime in this region is not clear. In region III, it is non-negative and the singularity located on the hypersurface  $x + x_0 = -2/3$  is naked, and the spacetime in this region can be considered as representing an inhomogeneous cosmological model. In region I, to study the nature of the singularity located on the hypersurface  $x + x_0 = 0$ , let us first calculate the gradient of the geometric radius,  $R \equiv rS(x)$ , of the two sphere,

$$R_{,\alpha}R_{,\beta}g^{\alpha\beta} = \frac{1}{9(x+x_0)^{2/3}} \left\{ 4r^2 - 9(x+x_0)^{2/3} \right\}. \quad (3.7)$$

The formation of apparent horizons are indicated by the vanishing of the gradient. Thus, setting the right-hand side of Eq.(3.7) to zero, we obtain

$$x + x_0 = \left(\frac{2r}{3}\right)^3, \quad (R_{,\alpha}R_{,\beta}g^{\alpha\beta} = 0). \quad (3.8)$$

Clearly, for any given  $r$ , we always have  $x + x_0 \geq 0$  on the apparent horizon. Hence, in the present case the formation of the spacetime singularity on the hypersurface  $x + x_0 = 0$  always follows the formation of the apparent horizon, or in other words, the singularity is always covered by the apparent horizon. These solutions can be considered as representing the formation of black holes due to the gravitational collapse of the fluid, starting at the moment  $t = -\infty$ . Defining the mass function  $m(t, r)$  as [14],

$$m(t, r) = \frac{R}{2} \left(1 + R_{,\alpha}R_{,\beta}g^{\alpha\beta}\right), \quad (3.9)$$

we find that  $m(t, r) = 2r^3/9$ . On the apparent horizon, we have

$$M_{BH}(t) = m(t, r_{AH}(t)) = \frac{2r_{AH}^3(t)}{9}, \quad (3.10)$$

where  $r_{AH}(t)$  is a solution of Eq.(3.8). The quantity  $M_{BH}(t)$  can be considered as the contribution of the collapsing perfect fluid to the total mass of such formed black holes [3], which in the present case goes to infinity as  $r_{AH}(t) \rightarrow +\infty$ , as can be seen from Eq.(3.8). Therefore, now the collapse of the dust fluid always forms black holes with infinitely large mass. To remedy this shortage, one may cut the spacetime along a non-spacelike hypersurface, say,  $r = r_0(t)$ , and then join the region,  $r \leq r_0(t)$ , to an asymptotically flat region [15]. By this way, we can see that the resultant model will represent the collapse of a dust ball with its radius  $r = r_0(t)$  [cf. Fig. 2]. From the moment  $t = t_f$  on, the ball collapses completely inside the apparent horizon, and the contribution of the collapsing ball to the total mass of such a formed black hole is given by

$$M_{BH} = M_{BH}(t_f), \quad (3.11)$$

where  $t_f$  is a solution of the equation,

$$r_0(t_f) = r_{AH}(t_f). \quad (3.12)$$

In the present case, since the fluid is co-moving with the coordinates, without loss of generality, we can choose the joining hypersurface as  $r_0(t) = r_0 = \text{Const.}$  Then, from Eq.(3.11) we find that  $M_{BH}(= 2r_1^3/9)$  is always finite and different from zero for any given non-zero  $r_0$ . This is different from the gravitational collapse of dust fluid with self-similarity of the first kind studied in [16], where it was shown that for any given non-zero  $r_0$ , black holes with infinitesimal mass can be formed by properly choosing a parameter that characterizes the strength of the collapse.

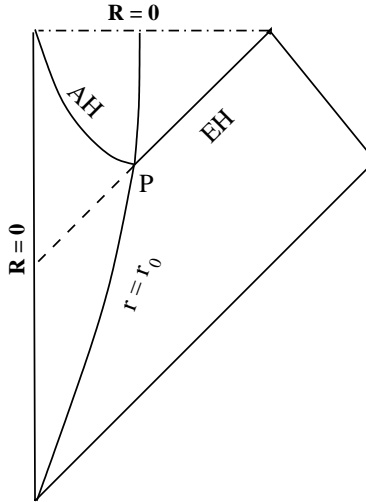


FIG. 2. The corresponding Penrose diagram of the spacetime described by Eq.(3.5), after it is first cut along the hypersurface  $r = r_0$  and then joined with an asymptotically flat region. The point  $P$  represents the moment when  $r_0 = r_{AH}(t_f)$ , where the ball of the fluid collapses completely inside the apparent horizon.

### B. Case $p_0 > 0$

In this case it can be shown that Eq.(3.3) has the solution,

$$y(x) = -a \tan A(x), \quad S(x) = \cos^{2/3} A(x), \quad (3.13)$$

where  $x_0$  is another integration constant, and

$$A(x) \equiv \frac{3a}{2} (x + x_0), \quad a \equiv \left| \frac{p_0}{3} \right|^{1/2}, \quad (3.14)$$

Then, from Eq.(3.4) we find that

$$\rho = \frac{3a^2 \tan A(x) [\tan A(x) - a]}{r_1^2 [1 - a \tan A(x)]}, \quad p = \frac{p_0}{r_1^2} > 0, \quad (3.15)$$

from which we can see that the solutions are singular on the hypersurfaces,

$$x + x_0 = \begin{cases} \frac{2}{3a} \left[ \tan^{-1} \left( \frac{1}{a} \right) + n\pi \right], & (a \neq 1), \\ \frac{2\pi}{3a} \left( n + \frac{1}{2} \right) \end{cases} \quad (3.16)$$

except for the case  $a = 1$ , where we have

$$\rho = -\frac{3}{r_1^2} \tan \left\{ \frac{3}{2} (x + x_0) \right\}, \quad (a = 1), \quad (3.17)$$

where  $n$  is an integer. Clearly, in the latter case the spacetime is singular only on the hypersurfaces

$$x + x_0 = \frac{2\pi}{3} \left( n + \frac{1}{2} \right), \quad (a = 1). \quad (3.18)$$

However, in either of the two cases, the spacetime is singular on various hypersurfaces, and the energy conditions, weak, strong and dominant [17], hold only in certain regions. The physics of these singularities are not clear, and the solutions may have physical applications only in certain regions. In particular, they cannot be interpreted as representing gravitational collapse of the fluid.

### C. Case $p_0 < 0$

In this case, the solutions can be further classified into three different cases, according to  $\alpha) y^2 > a^2$ ,  $\beta) y^2 = a^2$ , and  $\gamma) y^2 < a^2$ .

$\alpha)$  **Case**  $y^2 > a^2$ : In this case it can be show that Eq.(3.3) has the solution,

$$y(x) = a \frac{\cosh A(x)}{\sinh A(x)}, \quad S(x) = \sinh^{2/3} A(x), \quad (3.19)$$

while Eqs.(3.4) yields,

$$\begin{aligned} \rho &= \frac{3a^2 \cosh A(x) [\cosh A(x) + a \sinh A(x)]}{r_1^2 \sinh A(x) [\sinh A(x) + a \cosh A(x)]}, \\ p &= -\frac{|p_0|}{r_1^2} = -\frac{3a^2}{r_1^2}, \end{aligned} \quad (3.20)$$

where  $A(x)$  and  $a$  are still given by Eq.(3.14). The three energy conditions now require  $\rho \geq 9a^2$ . Then, from Eq.(3.20) we can see that this condition holds only in the region

$$\ln r + x_0 - x_1 \leq t \leq \ln r + x_0, \quad (3.21)$$

for any given  $a$ , or in the region

$$t \leq \ln r + x_0 + x_2, \quad (a < 1), \quad (3.22)$$

for  $a < 1$ , where

$$\begin{aligned} x_1 &\equiv \frac{1}{3a} \ln \left( \frac{2 + \sqrt{3 + a^2}}{1 + a} \right) > 0, \\ x_2 &\equiv -\frac{1}{3a} \ln \left| \frac{2 - \sqrt{3 + a^2}}{1 + a} \right| > 0. \end{aligned} \quad (3.23)$$

In the region defined by Eq.(3.21), the spacetime is limited by the curvature singularity located at  $x + x_0 = 0$  in one side, and by the hypersurface  $x + x_0 = x_1$  in the other side, across the latter the energy conditions do not hold. The solutions in this region seem not to have much physics. In the region defined by Eq.(3.22), the spacetime may be considered as representing a cosmological model. It is interesting to note that the spacetime in this region is free of singularities and asymptotically flat as  $t \rightarrow +\infty$ . However, it is not geodesically complete and needs to be extended beyond the hypersurface  $x + x_0 = -x_2$ . A “natural” extension would be the one simply given by the above solutions (3.19). This extension will be valid until the hypersurface  $x + x_0 = -x_3$ , where

$$x_3 \equiv -\frac{1}{3a} \ln \left| \frac{1 - a}{1 + a} \right|, \quad (3.24)$$

on which the spacetime is singular. Obviously, the fluid in this extended region do not satisfy all the three energy conditions.

$\beta)$  **Case**  $y^2 = a^2$ : In this case, it can be shown that Eq.(3.3) has the solution,

$$y(x) = \pm a, \quad S(x) = e^{\pm ax}, \quad \rho = -p = \frac{3a^2}{r_1^2}. \quad (3.25)$$

Introducing a new radial coordinate  $\bar{r}$  via the relation,  $\bar{r} = r^{1 \pm a}$ , we find that the corresponding metric can be written in the form,

$$ds^2 = r_1^2 \left\{ dt^2 - e^{\pm 2a(x_0 - t)} \left( d\bar{r}^2 + \bar{r}^2 d^2\Omega \right) \right\}, \quad (3.26)$$

which is the de Sitter solution [17].

$\gamma)$  **Case**  $y^2 < a^2$ : In this case, we find that

$$\begin{aligned} y(x) &= a \frac{\sinh A(x)}{\cosh A(x)}, \quad S(x) = \cosh^{2/3} A(x), \\ \rho &= \frac{3a^2 \sinh A(x) [\sinh A(x) + a \cosh A(x)]}{r_1^2 \cosh A(x) [\cosh A(x) + a \sinh A(x)]}, \\ p &= -\frac{3a^2}{r_1^2}. \end{aligned} \quad (3.27)$$

It can be shown that now the energy conditions hold only in the region,

$$\ln r + x_0 + x_3 \leq t \leq \ln r + x_0 + x_4, \quad (a \geq 1), \quad (3.28)$$

for  $a \geq 1$ , where  $x_3$  is given by Eq.(3.24) and  $x_4$  is given by

$$x_4 \equiv -\frac{1}{3a} \ln \left| \frac{a-1}{2 + \sqrt{3+a^2}} \right|. \quad (3.29)$$

When  $a < 1$ , there does not exist any region in which the three energy conditions hold. The spacetime is singular on the hypersurface  $x + x_0 = -x_3$ . The physics of the spacetime in this case is not clear (if there is any).

#### IV. ANALYTIC SOLUTIONS OF PERFECT FLUID WITH SELF-SIMILARITY OF THE FIRST AND SECOND KINDS AND THEIR PHYSICAL INTERPRETATIONS

The solutions to be studied in this section are those given by Eqs.(2.27) - (2.31) in [9], for which we have  $\Phi = 0$ . From Eq.(A.45) we find that the function  $\Psi$  takes the same form as that given by Eq.(3.1) in terms of  $y$ , while Eq.(A.44) yields,

$$y_{,xx} + (3y + \alpha)y_{,x} = 0, \quad (4.1)$$

which allows the first integral,

$$2y_{,x} + 3y^2 + 2\alpha y + \alpha^2 p_0 = 0, \quad (4.2)$$

where  $p_0$  is also a *dimensionless* constant, in contrast to what claimed in [9]. It can be shown that in the present case Eq.(A.39) is satisfied automatically, too. Then, the corresponding perfect fluid is given by

$$\rho = \frac{y[(3 - 2\alpha)y - \alpha^2 p_0]}{\alpha^2 r_1^2 (1 + y)t^2}, \quad p = \frac{p_0}{r_1^2 t^2}, \quad u_\mu = r_1 \delta_\mu^t. \quad (4.3)$$

When  $\alpha = 1$ , the corresponding solutions have self-similarity of the first kind, otherwise, they have the second kind.

Note that the solutions of Eq.(4.2) for the function  $y(x)$  given in [9] are not correct. Thus, in the following we shall first derive the correct expressions for  $y(x)$  and  $S(x)$ , and then study the physics of the solutions. Depending on the value of  $p_0$ , the solutions of Eq.(4.2) can be divided into several classes. In the following let us consider them one by one.

##### A. Case $p_0 = 0$

When  $p_0 = 0$ , Eq.(4.2) has the solution,

$$\begin{aligned}
y(x) &= \frac{2\alpha}{3[e^{\alpha(x+x_0)} - 1]}, \\
S(x) &= e^{-\frac{2\alpha}{3}(x+x_0)} [e^{\alpha(x+x_0)} - 1]^{2/3},
\end{aligned} \tag{4.4}$$

where  $x_0$  is another integration constant. The corresponding energy density of the fluid is given by

$$\rho = \frac{4(3-2\alpha)}{9r_1^2} \left\{ t^2 [e^{\alpha(x+x_0)} - 1] \left[ e^{\alpha(x+x_0)} + \frac{2\alpha-3}{3} \right] \right\}^{-1}. \tag{4.5}$$

Since  $p = 0$  in the present case, the above solutions must also belong to the general Tolmann-Bondi solutions [13].

**a) Case  $0 < \alpha < 1$ :** In this subcase, Eq.(4.5) shows that the spacetime is singular on the hypersurfaces,

$$a) \ t = 0, \quad b) \ x + x_0 = 0, \quad c) \ x + x_0 = -x_5, \tag{4.6}$$

where  $x_5$  is defined as

$$x_5 \equiv -\frac{1}{\alpha} \ln \left| \frac{3-2\alpha}{3} \right| > 0. \tag{4.7}$$

From the above we can show that  $\rho$  is non-negative only in the region  $x + x_0 \leq -x_5$  or in the region  $x + x_0 \geq 0$ . In the region  $x + x_0 \geq 0$ , the spacetime is singular on the two hypersurfaces  $x + x_0 = 0$  and  $t = 0$ . While the physical meaning of the spacetime in the region  $x + x_0 \geq 0$ , and  $t \leq 0$  is not clear, the spacetime in the region  $t \geq 0$  can be considered as representing a cosmological model with its initial singularity at  $t = 0$ . The spacetime in the region  $x + x_0 \leq -x_5$  can be considered as representing the gravitational collapse of the perfect fluid. To study the nature of the spacetime singularity at  $x + x_0 = -x_5$ , let us consider the quantity,

$$R_{,\alpha} R_{,\beta} g^{\alpha\beta} = \frac{4\alpha^2 e^{2[(1-\alpha)x - \alpha x_0]}}{9(-t)^{2(\alpha-1)/\alpha} [e^{-\alpha(x+x_0)} - 1]^{2/3}} - 1, \tag{4.8}$$

from which we find that

$$(-t_{AH})^{\frac{\alpha-1}{\alpha}} = \frac{2\alpha e^{(1-\alpha)x - \alpha x_0}}{3[e^{-\alpha(x+x_0)} - 1]^{1/3}}, \quad (R_{,\alpha} R_{,\beta} g^{\alpha\beta} = 0). \tag{4.9}$$



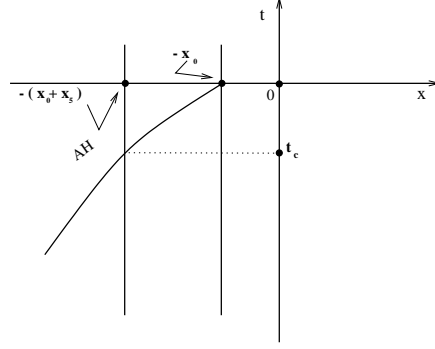


FIG. 3. The spacetime described by the solutions of Eq.(4.4) for  $0 < \alpha < 1$  in the  $(t, x)$ -plane. It is singular on the hypersurfaces a)  $t = 0$ , b)  $x = -x_0$  and c)  $x = -(x_0 + x_5)$ . At the moment  $t = t_c$  the apparent horizon crosses the singular hypersurface  $x = -(x_0 + x_5)$ , and asymptotically approaches to the point  $(t, x) = (0, -x_0)$ .

It can be shown that this hypersurface will cross the singular hypersurface  $x + x_0 = -x_5$  at the moment  $t = t_c$ , where  $t_c$  is given by

$$(-t_c)^{\frac{1-\alpha}{\alpha}} = \frac{3}{2a} [e^{\alpha x_5} - 1]^{1/3} e^{(1-\alpha)x_5 + x_0}. \quad (4.10)$$

This can be seen clearly in the  $(t, x)$ -plane, as illustrated by Fig. 3. The corresponding Penrose diagram is given by Fig. 4, from which we can see that the spacetime singularity formed at  $x + x_0 = -x_5$  is covered by the apparent horizon at the beginning ( $t < t_c$ ). As the fluid continues to collapse, the apparent horizon starts to form after the formation of the spacetime singularity, hence it becomes naked when  $t > t_c$ .

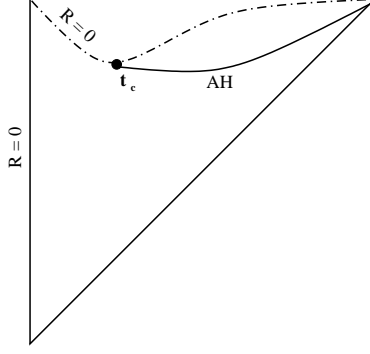


FIG. 4. The Penrose diagram for the spacetime described by the solutions of Eq.(4.4) for  $0 < \alpha < 1$ . The singularity formed on the hypersurface  $x + x_0 = -x_5$  is covered by the apparent horizon when  $t < t_c$  and becomes naked when  $t > t_c$ , where  $t_c$  is defined by Eq.(4.10).

**b) Case  $\alpha = 1$ :** In this subcase, it can be shown that the apparent horizon is given by

$$x + x_0 = -x_6, \quad (R_{,\alpha} R_{,\beta} g^{\alpha\beta} = 0), \quad (4.11)$$

where

$$x_6 \equiv \ln \left\{ 1 + \frac{8}{27} e^{-3x_0} \right\}. \quad (4.12)$$

Since

$$x_6 - x_5 = \ln \left( \frac{27 + 8e^{-3x_0}}{81} \right) = \begin{cases} > 0, & x_0 < p^*, \\ = 0, & x_0 = p^*, \\ < 0, & x_0 > p^*, \end{cases} \quad (4.13)$$

where  $p^* \equiv -[\ln(27/4)]/3$ , we find that, when  $x_0 < p^*$ , the apparent horizon always forms before the formation of the spacetime singularity at  $x + x_0 = -x_5$ , that is, the collapse now always forms black holes. When  $x_0 = p^*$ , the apparent horizon and the spacetime singularity are formed on the same hypersurface, i.e., now the singularity is marginally naked. When  $x_0 > p^*$ , the apparent horizon always forms after the formation of the spacetime singularity, or in other words, now the collapse always forms naked singularities. The contribution of the collapsing fluid to the total mass of such formed black holes is given by

$$M_{BH}(t) = \frac{2e^{-2x_0}}{9}r_{AH}(t), \quad (4.14)$$

where  $r_{AH}(t)$  is a solution of Eq.(4.11). Thus, as  $r_{AH}(t) \rightarrow +\infty$ , we find  $M_{BH} \rightarrow +\infty$ . Similar to the case discussed in the last section, to obtain a black hole with finite mass, we can cut the spacetime along the hypersurface  $r = r_0 = Const.$  and then join the region  $r \leq r_0$  with an asymptotically flat region. By this way, the resulting model will represent gravitational collapse of a ball with its comoving radius  $r_0$ . At the moment  $t = t_f$ , where  $t_f$  is a solution of the equation  $r_{AH}(t_f) = r_0$ , the ball collapses completely inside the horizon, and its contribution to the total mass of such formed black holes is given by

$$M_{BH} = \frac{2e^{-2x_0}}{9}r_0, \quad (4.15)$$

which is always finite and non-zero for any given non-zero  $r_0$ . It is interesting to note that in the present case the solutions may represent critical phenomena. To have a definite answer to this problem, we need to study the spectrum of perturbations of the “critical” solution and show that it has only one unstable mode. This is currently under our investigation.

It is very interesting to note that in this case the black holes start to form with a mass gap, although the Einstein field equations are of scale invariance, and the spacetime has self-similarity of the first kind [3,4]. Thus, the solutions studied in the present case show clearly that even the solutions have homothetic self-similarity, the formation of black holes does not necessarily start with an infinitesimal mass.

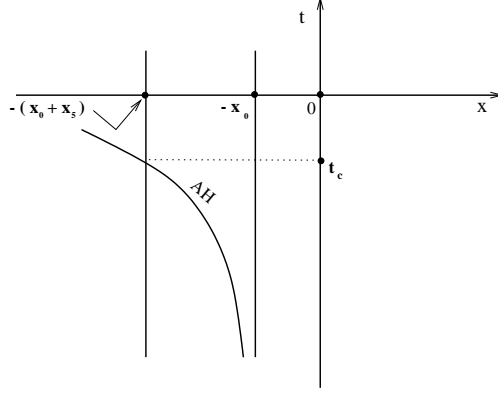


FIG. 5. The spacetime described by the solutions of Eq.(4.4) for  $1 < \alpha < 3/2$  in the  $(t, x)$ -plane. It is singular on the hypersurfaces a)  $t = 0$ , b)  $x = -x_0$  and c)  $x = -(x_0 + x_5)$ . At the moment  $t = t_c$  the apparent horizon crosses the singular hypersurface  $x = -(x_0 + x_5)$ , and asymptotically approaches to the one  $x = -x_0$ .

**c) Case  $1 < \alpha < \frac{3}{2}$ :** In this subcase, it can be shown that the spacetime is also singular on the hypersurfaces given by Eq.(4.6) and the apparent horizon is given by Eq.(4.9). In the  $(t, x)$ -plane, it is given by Fig. 5, from which we can see that it also crosses the singular hypersurface  $x + x_0 = -x_5$  once, but in contrast to the subcase  $0 < \alpha < 1$ , now the singularity initially is naked and becomes covered by the apparent horizon after the moment  $t = t_c$ , as shown by Fig. 6.

**d) Case  $\alpha = \frac{3}{2}$ :** From Eq.(4.5) we find that  $\rho = 0 = p$ . That is, in this subcase the spacetime is vacuum. The metric coefficients are given by

$$\begin{aligned} y(x) &= \left[ e^{3(x+x_0)/2} - 1 \right]^{-1}, \\ S(x) &= e^{-(x+x_0)} \left[ e^{3(x+x_0)/2} - 1 \right]^{2/3}. \end{aligned} \quad (4.16)$$

Defining a new radial coordinate  $\tilde{r}$  via the relations,

$$\tilde{r} \equiv \frac{2}{3} e^{3x_0/2} r^{3/2}, \quad (4.17)$$

we find that the metric can be written as

$$ds^2 = d\tilde{\tau}^2 - r_g^{2/3} \left\{ \frac{d\tilde{r}^2}{\left[\frac{3}{2}(\tilde{r} - \tilde{\tau})\right]^{2/3}} - \left[\frac{3}{2}(\tilde{r} - \tilde{\tau})\right]^{4/3} d^2\Omega \right\}, \quad (4.18)$$

where

$$\tilde{\tau} = -t, \quad r_g = e^{-3x_0}. \quad (4.19)$$

This is exactly the Schwarzschild solution written in the Lemaitre coordinates [18], with  $r_g$  being the Schwarzschild radius. From this case we can see that the parameter  $x_0$  is related to the total mass of the Schwarzschild black hole.

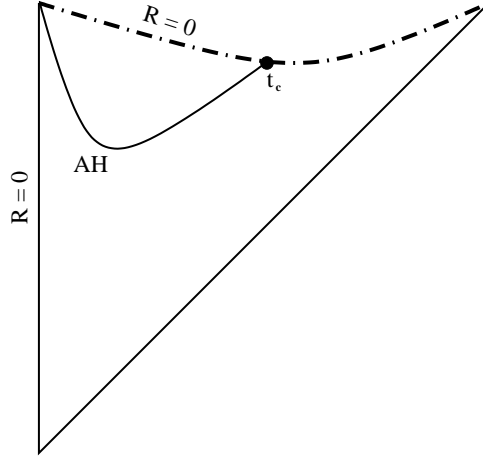


FIG. 6. The Penrose diagram for the spacetime described by the solutions of Eq.(4.4) for  $1 < \alpha < 3/2$ . The singularity formed on the hypersurface  $x + x_0 = -x_5$  is naked for  $t < t_c$  and covered by the apparent horizon when  $t > t_c$ , where  $t_c$  is defined by Eq.(4.10).

**e) Case  $\alpha > \frac{3}{2}$ :** In this subcase the energy density of the fluid takes the form

$$\rho = \frac{4(2\alpha - 3)}{9r_1^2} \left\{ t^2 \left[ 1 - e^{\alpha(x+x_0)} \right] \left[ e^{\alpha(x+x_0)} + \frac{2\alpha - 3}{3} \right] \right\}^{-1}. \quad (4.20)$$

Thus,  $\rho \geq 0$  requires

$$x + x_0 \leq 0, \quad (\rho \geq 0). \quad (4.21)$$

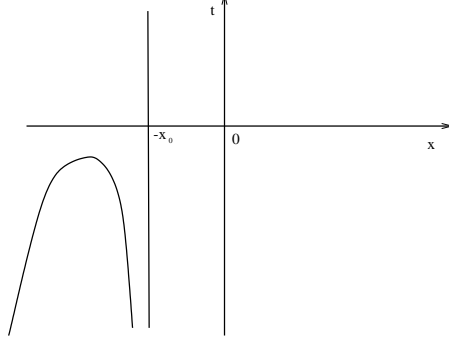


FIG. 7. The spacetime described by the solutions of Eq.(4.4) for  $\alpha > 3/2$  in the  $(t, x)$ -plane. It is singular on the hypersurface  $x = -x_0$  and the energy density of the fluid is non-negative only in the region  $x \leq -x_0$ .

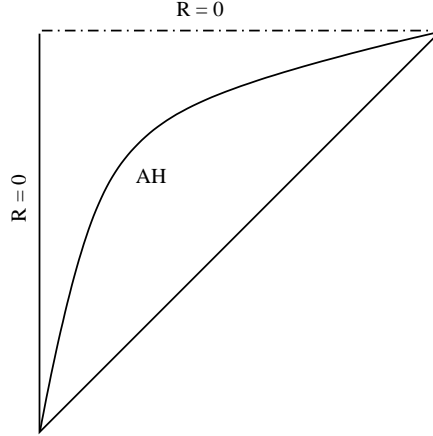


FIG. 8. The Penrose diagram for the spacetime described by the solutions of Eq.(4.4) for  $\alpha > 3/2$ . The apparent horizon now always forms before the formation of the spacetime singularity at  $x + x_0 = 0$ .

In this region the spacetime is singular only on the hypersurface  $x + x_0 = 0$ . On the other hand, the apparent horizon in the present case is still given by Eq.(4.9). In the  $(t, x)$ -plane, this hypersurface is shown by Fig. 7. The corresponding Penrose diagram is given by Fig. 8, from which we can see that now the apparent horizon always forms before the formation of the spacetime singularity, that is, the solutions now represent the formation of black holes.

The contribution of the fluid to the total mass of such formed black holes is given by

$$M_{BH}(t) = \frac{r_{AH}(t)}{2} \left[ e^{-\alpha(x_{AH}+x_0)} - 1 \right]^{2/3}, \quad (4.22)$$

where  $r_{AH}$  and  $x_{AH}$  are the solution of Eq.(4.9). When the spacetime is first cut along the hypersurface  $r = r_0$  and then joined with an asymptotically flat region, and the contribution of the fluid to the total mass of such formed black holes is given by

$$M_{BH} = \frac{r_0}{2} \left[ e^{-\alpha(x_f+x_0)} - 1 \right]^{2/3}, \quad (4.23)$$

where  $x_f = \ln[r_0/(-t_f)^{1/\alpha}]$ , and  $t_f$  denotes the moment when the ball collapses completely inside the apparent horizon, given by  $r_0 = r_{AH}(t_f)$ . One can show that this mass is also finite and non-zero for any given non-zero  $r_0$ .

### B. Case $0 < p_0 < \frac{1}{3}$

In this case, it can be shown that the corresponding solutions are given by

$$\begin{aligned} y(x) &= \frac{1}{3} \left\{ \beta \tanh \left[ \frac{\beta}{2}(x + x_0) \right] - \alpha \right\}, \\ S(x) &= e^{-\alpha x/3} \cosh^{2/3} \left[ \frac{\beta}{2}(x + x_0) \right], \end{aligned} \quad (4.24)$$

where  $\beta \equiv \alpha|1 - 3p_0|^{1/2}$ . Then, the energy density of the fluid is given by

$$\begin{aligned} \rho &= \frac{\beta(2\alpha - 3) \left\{ (1 - 3p_0)^{-1/2} - \tanh \left[ \frac{\beta}{2}(x + x_0) \right] \right\}}{3\alpha^2 r_1^2 t^2 \left\{ \tanh \left[ \frac{\beta}{2}(x + x_0) \right] + A \right\}} \\ &\quad \times \left\{ \tanh \left[ \frac{\beta}{2}(x + x_0) \right] + B \right\}, \end{aligned} \quad (4.25)$$

where

$$A \equiv \frac{3 - \alpha}{\alpha |1 - 3p_0|^{1/2}}, \quad B \equiv \frac{3\alpha p_0 - (2\alpha - 3)}{|1 - 3p_0|^{1/2} (2\alpha - 3)}. \quad (4.26)$$

From the above equation it can be shown that, when  $0 < \alpha < \alpha_1$ , we have  $A > 1$ ; when  $\alpha_1 \leq \alpha \leq \alpha_2$ , we have  $-1 \leq A \leq +1$ ; and when  $\alpha > \alpha_2$ , we have  $A < -1$ , where

$$\alpha_1 \equiv \frac{3}{1 + (1 - 3p_0)^{1/2}}, \quad \alpha_2 \equiv \frac{3}{1 - (1 - 3p_0)^{1/2}}. \quad (4.27)$$

When  $0 < \alpha \leq 3/2$ , we have  $B < -(1 - 3p_0)^{-1/2} < -1$ ; when  $3/2 < \alpha < \alpha_1$ , we have  $B > +1$ ; when  $\alpha_1 \leq \alpha \leq \alpha_2$ , we have  $-1 \leq B \leq +1$ ; and when  $\alpha > \alpha_2$ , we have  $B < -1$  [cf. Fig. 9]. Thus, from Eq.(4.25) we find that  $\rho$  is always non-negative at any given point of the spacetime when  $0 < \alpha < \alpha_1$  or when  $\alpha > \alpha_2$ . It is singular on the hypersurface  $t = 0$ . This hypersurface divides the spacetime into two disconnected regions,  $t \geq 0$  and  $t \leq 0$ . In the region  $t \geq 0$ , the spacetime can be considered as representing a cosmological model with its initial singularity at  $t = 0$ . In the region  $t \leq 0$ , the spacetime can be interpreted as representing the gravitational collapse of the perfect fluid. In this region, an apparent horizon is formed on the hypersurface, given by

$$(-t_{AH})^{\frac{\alpha-1}{\alpha}} = \frac{\alpha e^{(3-\alpha)x/3}}{3 \cosh^{1/3} \left[ \frac{\beta}{2}(x + x_0) \right]} \times \left\{ \cosh \left[ \frac{\beta}{2}(x + x_0) \right] - (1 - 3p_0)^{1/2} \right\}, \quad (R_{,\alpha} R_{,\beta} g^{\alpha\beta} = 0). \quad (4.28)$$



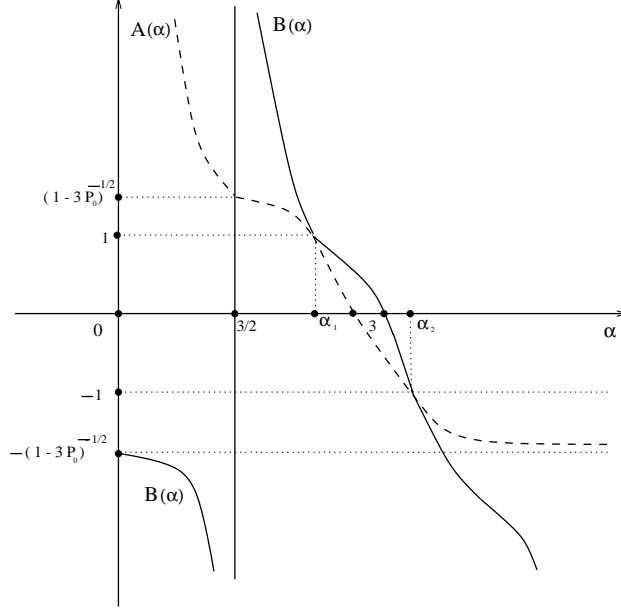


FIG. 9. The curves of the functions  $A(\alpha)$  and  $B(\alpha)$ , defined by Eq.(4.26) versus  $\alpha$ .

It is not difficult to see that this hypersurface is formed always before the formation of the spacetime singularity at  $t = 0$ . Thus in the present case the collapse always forms black holes, and the contribution of the fluid to the black hole mass is given by

$$M_{BH}(t) = \frac{r_{AH}(t)}{2} \cosh^{2/3} \left[ \frac{3}{2}(x_{AH} + x_0) \right] e^{-\alpha x_{AH}/3}, \quad (4.29)$$

where  $r_{AH}(t)$  and  $x_{AH}$  are the solutions of Eq.(4.28). Similar to the cases discussed above, the mass of such formed black holes now becomes also infinitely large as  $r_{AH}(t) \rightarrow +\infty$ . Thus, in this case we also need to cut the spacetime along the hypersurface  $r = r_0$  and then join the region  $r \leq r_0$  to an asymptotically flat region. Once this is done, it is not difficult to see that

$$M_{BH} = \frac{r_0}{2} \cosh^{2/3} \left[ \frac{3}{2}(x_f + x_0) \right] e^{-\alpha x_f/3}, \quad (4.30)$$

where  $x_f$  is given by  $x_f = \ln[r_0(-t_f)^{-1/\alpha}]$ , and  $t_f$  denotes the moment when the ball of perfect fluid collapses completely inside the apparent horizon, which is given by Eq.(4.28)

with  $r_{AH}(t_f) = r_0$ . Clearly, for any given non-zero  $r_0$ ,  $M_{BH}$  is non-zero. That is, in the present case the black holes start to form with a mass gap, too.

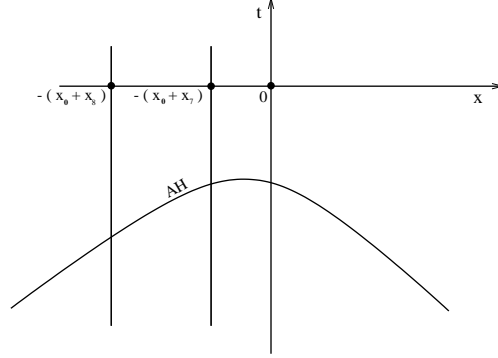


FIG. 10. The spacetime described by the solutions of Eq.(4.24) for  $\alpha_1 \leq \alpha \leq \alpha_2$  in the  $(t, x)$ -plane.  $\rho$  is non-negative only in the region  $x \geq -(x_0 + x_7)$  or in the region  $x \leq -(x_0 + x_8)$ . It is singular on the hypersurface  $x = -(x_0 + x_7)$ .

When  $\alpha_1 \leq \alpha \leq \alpha_2$ ,  $\rho$  is non-negative only in the region  $x + x_0 \geq -x_7$  or in the region  $x + x_0 \leq -x_8$ , where

$$x_7 \equiv \frac{2}{\beta} \tanh^{-1}(A), \quad x_8 \equiv \frac{2}{\beta} \tanh^{-1}(B). \quad (4.31)$$

Since  $B \geq A$  for  $\alpha_1 \leq \alpha \leq \alpha_2$ , we find  $x_8 \geq x_7$ , where equality holds only when  $\alpha = \alpha_1$ , or  $\alpha = \alpha_2$ . The spacetime is singular on the hypersurface  $t = 0$  and  $x + x_0 = -x_7$ . Once again, the region  $t \geq 0$  can be considered as representing a cosmological model. In the region  $x + x_0 \leq -x_7$ , on the other hand, we find that

$$\rho = \begin{cases} -\infty, & x + x_0 = -x_7, \\ < 0, & -x_8 < x + x_0 < -x_7, \\ = 0, & x + x_0 = -x_8, \\ > 0, & x + x_0 < -x_8. \end{cases} \quad (4.32)$$

In this region, the apparent horizon is still given by Eq.(4.28), from which we find that, as  $x \rightarrow \pm\infty$ , we have  $(-t)^{(\alpha-1)/\alpha} \rightarrow +\infty$ . In the  $(t, x)$ -plane, it is given by a curve that crosses

both the hypersurfaces  $x + x_0 = -x_7$  and  $x + x_0 = -x_8$  [cf. Fig. 10]. The corresponding Penrose diagram is given by Fig. 11, the physics of which is unclear.

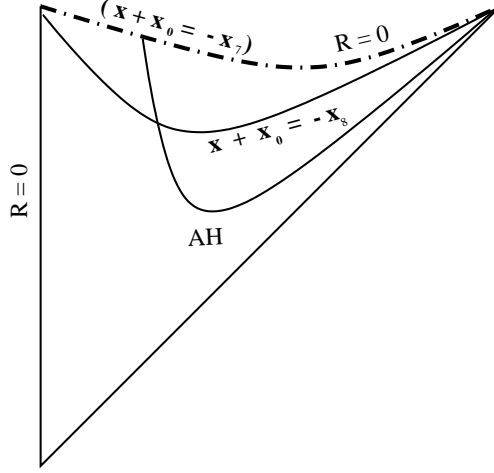


FIG. 11. The Penrose diagram for the spacetime described by the solutions of Eq.(4.24) for  $\alpha_1 \leq \alpha \leq \alpha_2$ . In the region  $-x_7 \leq x + x_0 \leq -x_8$ , the energy density of the fluid becomes negative.

### C. Case $p_0 = \frac{1}{3}$

In this case, it can be shown that Eqs.(4.2) and (4.3) have the following solutions,

$$\begin{aligned}
 y &= \frac{2}{3(x+x_0)} - \frac{\alpha}{3}, \quad S = (x+x_0)^{2/3} e^{-\alpha x/3}, \\
 \rho &= \frac{1}{3\alpha^2 r_1^2 t^2 (x+x_0) [(3-\alpha)(x+x_0) + 2]} \\
 &\quad \times \left\{ \alpha^2 (3-\alpha)(x+x_0)^2 \right. \\
 &\quad \left. - 6\alpha(2-\alpha)(x+x_0) + 4(3-2\alpha) \right\}, \\
 p &= \frac{1}{3r_1^2 t^2}. \tag{4.33}
 \end{aligned}$$

**A) Case  $0 < \alpha < 1$ :** In this subcase from Eq.(4.33) we can see that the spacetime is singular on the hypersurfaces

$$a) \ t = 0, \quad b) \ x + x_0 = 0, \quad c) \ x + x_0 = -\frac{2}{3-\alpha}. \quad (4.34)$$

It is not difficult to show that now the singularity at  $x + x_0 = -2/(3-\alpha)$  is first formed and the one at  $t = 0$  is last formed. The spacetime in the region  $t \geq 0$  may be considered as representing cosmological model with its initial singularity of the spacetime at  $t = 0$ . The region  $x + x_0 \leq -2/(3-\alpha)$  can be considered as representing the gravitational collapse of the perfect fluid. To study the nature of this singularity, let us consider the formation of apparent horizons, given by

$$(-t_{AH})^{\frac{2(1-\alpha)}{\alpha}} = \frac{9(x+x_0)^{2/3}}{[2-\alpha(x+x_0)]^2} e^{-\frac{2(3-\alpha)}{3}x}, \quad (R_{,\alpha}R_{,\beta}g^{\alpha\beta} = 0). \quad (4.35)$$

In the  $(t, x)$ -plane, this curve is similar to that given in Fig. 3, if  $x_5$  is replaced by  $2/(3-\alpha)$ . Thus, in this case the singularity at  $x + x_0 = -2/(3-\alpha)$  is covered upto the moment  $t = t_c$ , where  $t_c$  now is given by

$$(-t_c)^{\frac{(1-\alpha)}{\alpha}} = \frac{1}{3} \left( \frac{3-\alpha}{2} \right)^{2/3} e^{2/3}. \quad (4.36)$$

After this moment, the singularity becomes naked. The corresponding Penrose diagram is given by Fig. 4.

**B) Case  $\alpha = 1$ :** As we mentioned previously, when  $\alpha = 1$ , the corresponding solutions have the self-similarity of the first kind. Setting  $\alpha = 1$  in the above expressions, we find that

$$\rho = \frac{(x+x_0)^2 - 3(x+x_0) + 2}{3r_1^2 t^2 (x+x_0) [(x+x_0) + 1]}, \quad p = \frac{1}{3r_1^2 t^2}. \quad (4.37)$$

Clearly, the spacetime is also singular on the hypersurfaces given by Eq.(4.34). Since now we have  $e^x = r/(-t)$ , we can see that these singular hypersurfaces are straight lines in the  $(t, r)$ -plane. The spacetime in the region  $t \geq 0$  may be interpreted as representing cosmological model with an initial spacetime singularity at  $t = 0$ . The region  $x + x_0 \leq -1$  may be considered as representing the gravitational collapse of the perfect fluid starting at  $t = -\infty$ . To study the nature of the singularity located on the hypersurface  $x + x_0 = -1$ ,

let us, following the analysis given above, first consider the formation of apparent horizon in this region,

$$R_{,\alpha}R_{,\beta}g^{\alpha\beta} = \frac{e^{4x/3}}{9(x+x_0)^{2/3}} \{Y_1(x) - Y_2(x)\}, \quad (4.38)$$

where

$$Y_1(x) \equiv [2 - (x + x_0)]^2, \quad Y_2(x) \equiv 9(x + x_0)^{2/3}e^{-4x/3}. \quad (4.39)$$

It is easy to show that  $Y_2(x)$  has one minimal at  $x + x_0 = 0$  and one maximal at  $x + x_0 = 1/2$ . When  $x \rightarrow -\infty$  it diverges exponentially, and when  $x \rightarrow +\infty$  it goes to zero exponentially. On the other hand,  $Y_1(x)$  is a parabola with its minimum located at  $x + x_0 = 2$  [cf. Fig. 12]. Thus, in general the equation  $Y_1(x) = Y_2(x)$  has three real roots, say,  $x_9$ ,  $x_{10}$  and  $x_{11}$ . Without loss of generality, we assume that  $x_{11} > x_{10} > x_9$ . Then, we can see that  $x = x_9$  represents the outmost trapped surface, i.e., the apparent horizon. Introducing a new parameter  $D$  via the relation,

$$D \equiv -(x_9 + x_0 + 1), \quad (4.40)$$

we find that the hypersurface  $x = x_9$  can be written as

$$x + x_0 = -(1 + D), \quad (R_{,\alpha}R_{,\beta}g^{\alpha\beta} = 0). \quad (4.41)$$

Thus, when  $D > 0$ , the apparent horizon always forms before the formation of the spacetime singularity at  $x + x_0 = -1$ . That is, in this case the gravitational collapse of the perfect fluid always forms black holes. When  $D < 0$ , the apparent horizon always forms after the formation of the spacetime singularity at  $x + x_0 = -1$ , namely, now the collapse always forms naked singularities. When  $D = 0$ , the apparent horizon and the spacetime singularity are formed on the same hypersurface  $x + x_0 = -1$ , and now the singularity is marginally naked. In the last case, it can be shown that

$$x_9 = x_0 = 0, \quad (D = 0), \quad (4.42)$$

and the corresponding solution is given by

$$S = x^{2/3}e^{-x/3}, \quad y = \frac{2-x}{3x}, \quad (D = 0). \quad (4.43)$$

Similar to the case  $p_0 = 0$  and  $\alpha = 1$  discussed above in this section, these solution may also represent critical collapse. To have a definite answer to this problem, we need to study perturbations of the “critical” solution, which is out of the scope of this paper, and we hope to return to this problem in another occasion.

On the other hand, the mass function defined by Eq.(3.9) now takes the form,

$$m(t, r) = \frac{rY_1(x)}{18}e^x. \quad (4.44)$$

Thus, on the apparent horizon  $x = x_9$ , we have

$$M_{BH}(t) = \frac{(3+D)^2e^{x_9}}{18}r_{AH}(t), \quad (4.45)$$

which shows that, as  $r_{AH}(t) \rightarrow +\infty$ , the total mass of black hole becomes infinitely large. Similar to the cases considered above, to have a black hole with finite mass, we can make a “surgery” to the spacetime. By this way, we can see that the resultant solution will represent a collapsing ball with a finite radius  $r_0$ , and its contribution to the total mass of black hole is given by

$$M_{BH} = \frac{(3+D)^2e^{x_9}}{18}r_0, \quad (D \geq 0), \quad (4.46)$$

which is finite and non-zero for any given non-zero  $r_0$ .

**C) Case  $1 < \alpha < 3$ :** In this case the spacetime singularities and apparent horizon are still given by Eqs.(4.34) and (4.35), respectively. It can be shown that the curve that represent the apparent horizon in the  $(t, x)$ -plane now is similar to that given in Fig. 5, that is, in the present case the singularity at  $x + x_0 = -2/(3 - \alpha)$  is initially naked. As the fluid is collapsing until the moment  $t_c$  given by Eq.(4.36), the apparent horizon starts to form. The corresponding Penrose diagram is given by Fig. 6.

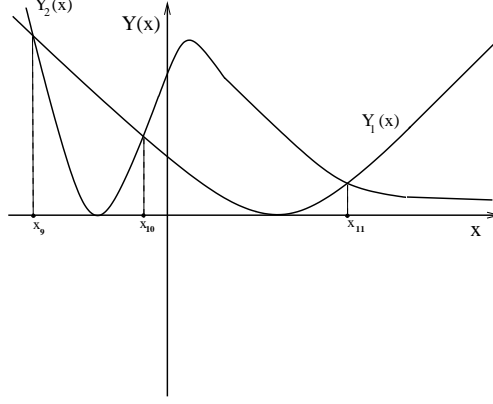


FIG. 12. The curves of the functions  $Y_1(x)$  and  $Y_2(x)$ , defined by Eq.(4.39) versus  $x$ . The equation  $Y_1(x) = Y_2(x)$  in general has three real roots,  $x_9$ ,  $x_{10}$  and  $x_{11}$ , with  $x_{11} > x_{10} > x_9$ .

**D) Case  $\alpha = 3$ :** In this subcase, the corresponding physical quantities are given by

$$\rho = \frac{3(x + x_0) - 2}{9r_1^2 t^2 (x + x_0)},$$

$$R_{,\alpha} R_{,\beta} g^{\alpha\beta} = \frac{[3(x + x_0) - 2]^2}{9(-3t)^{4/3} (x + x_0)^{2/3}} - 1, \quad (4.47)$$

which show that the spacetime is singular at

$$a) t = 0, \quad b) x + x_0 = 0. \quad (4.48)$$

The location of the apparent horizon is given by

$$(-t_{AH})^{4/3} = \frac{\left[(x + x_0) - \frac{2}{3}\right]^2}{(x + x_0)^{2/3}}, \quad (R_{,\alpha} R_{,\beta} g^{\alpha\beta} = 0), \quad (4.49)$$

from which we can see that in the  $(t, x)$ -plane it is given by the curved given in Fig. 7. Consequently, the solutions in this case represent gravitational collapse that always forms black holes.

**E) Case  $\alpha > 3$ :** In this case, following the same routine given above, it is not difficult to show that the spacetime singularity at  $x + x_0 = 0$ , formed due to the collapse of the perfect fluid, is also covered by an apparent horizon. It can be shown that in the last two subcases the mass of such formed black holes is always finite and non-zero.

### D. Case $p_0 > \frac{1}{3}$

In this case, the corresponding solutions are given by

$$\begin{aligned} y(x) &= -\frac{1}{3} \left\{ \beta \tan \left[ \frac{\beta}{2}(x + x_0) \right] + \alpha \right\}, \\ S(x) &= e^{-\alpha x/3} \cos^{2/3} \left[ \frac{\beta}{2}(x + x_0) \right], \end{aligned} \quad (4.50)$$

where  $\beta$  is given as that in Eq.(4.24). Then, the energy density of the fluid is given by

$$\begin{aligned} \rho &= \frac{\beta(2\alpha - 3) \left\{ (3p_0 - 1)^{-1/2} + \tan \left[ \frac{\beta}{2}(x + x_0) \right] \right\}}{3\alpha^2 r_1^2 t^2 \left\{ \tan \left[ \frac{\beta}{2}(x + x_0) \right] - A \right\}} \\ &\quad \times \left\{ \tan \left[ \frac{\beta}{2}(x + x_0) \right] - B \right\}, \end{aligned} \quad (4.51)$$

where  $A$  and  $B$  are given by Eq.(4.26). From the above expression we can see that  $\rho$  is non-negative only in certain regions and the spacetime is singular on various hypersurfaces. The solutions in this case cannot be interpreted as representing gravitational collapse of perfect fluid.

## V. CONCLUDING REMARKS

In Sec. III, we have studied the self-similar solutions of the zeroth kind, and found that some represent cosmological models and some represent gravitational collapse, while the others have no physical meanings. The ones that represent gravitational collapse are given by  $p = 0$ , i.e., dust fluid. These dust fluid solutions always collapse to form black holes with finite and non-zero mass.

In Sec. IV, self-similar solutions of both the first and the second kinds have been studied. In particular, it has been found that the self-similar solutions of the first kind ( $\alpha = 1$ ) with  $p_0 = 0$  or  $p_0 = 1/3$  may represent critical collapse but in the sense that now the “critical” solution separates solutions that form black holes to the solutions that form naked singularities. In this case the formation of black holes also starts with a mass gap. To show explicitly



that these solutions indeed represent critical collapse, the analysis of spectrum of perturbations of these “critical” solutions is needed, which are currently under our investigation. The solutions with  $p_0 = 0$  and  $\alpha > 3/2$ , the ones with  $0 < p_0 < 1/3$  and  $\alpha_1 < \alpha < \alpha_2$ , and the ones with  $p_0 = 1/3$  and  $\alpha \geq 3$  also represent gravitational collapse, and the collapse always forms black holes with finite and non-zero mass. The solutions with  $p_0 = 0$  and  $0 < \alpha < 1$  and the ones with  $p_0 = 1/3$  and  $0 < \alpha < 1$  represent the formations of spacetime singularities that are covered by apparent horizons at the beginning of the collapse and late become naked, while the ones with  $p_0 = 0$  and  $1 < \alpha < 3/2$  and the ones with  $p_0 = 1/3$  and  $1 < \alpha < 3$  represent the formations of spacetime singularities that are naked at the beginning of the collapse and late become covered by apparent horizons. All the rest of the solutions can be either considered as representing cosmological models with an initial spacetime singularity or have no physical meanings.

In review of all the above, one can see that the BC solutions with self-similarity of the zeroth and second kinds seem irrelevant to critical phenomena in gravitational collapse, and the only possible candidates for critical collapse are those solutions with self-similarity of the first (homothetic) kind with  $p_0 = 0$  or  $p_0 = 1/3$ , given in Sec.IV.

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## APPENDIX

The metric for spacetimes with spherical symmetry can be cost in the general form,

$$ds^2 = r_1^2 \left\{ e^{2\Phi(t,r)} dt^2 - e^{2\Psi(t,r)} dr^2 - r^2 S(t,r)^2 d\Omega^2 \right\}, \quad (\text{A.1})$$

where  $d\Omega^2 \equiv d\theta^2 + \sin^2(\theta) d\varphi^2$ , and  $r_1$  is a constant and has dimension of length,  $l$ . Then, it is easy to show that the coordinates  $\{x^\mu\} = \{t, r, \theta, \varphi\}$ , the Christoffel symbols,  $\Gamma_{\lambda\nu}^\mu$ , the Riemann tensor,  $R_{\mu\nu\lambda}^\sigma$ , the Ricci tensor,  $R_{\mu\nu}$ , and the Einstein tensor,  $G_{\mu\nu}$ , are all *dimensionless*, while the Ricci scalar,  $R$ , has the dimension of  $l^{-2}$ , and the Kretschmann scalar,  $I \equiv R^{\sigma\mu\nu\lambda} R_{\sigma\mu\nu\lambda}$ , has the dimension of  $l^{-4}$ .

For the metric (A.1), we find that the non-vanishing Christoffel symbols are given by

$$\begin{aligned} \Gamma_{00}^0 &= \Phi_{,t}, \quad \Gamma_{01}^0 = \Phi_{,r}, \quad \Gamma_{11}^0 = e^{2(\Psi-\Phi)} \Psi_{,t}, \\ \Gamma_{22}^0 &= r^2 S e^{-2\Phi} S_{,t}, \quad \Gamma_{33}^0 = r^2 S \sin^2 \theta e^{-2\Phi} S_{,t}, \\ \Gamma_{00}^1 &= e^{2(\Phi-\Psi)} \Phi_{,r}, \quad \Gamma_{01}^1 = \Psi_{,t}, \quad \Gamma_{11}^1 = \Psi_{,r}, \\ \Gamma_{22}^1 &= -r S e^{-2\Psi} (r S_{,r} + S), \quad \Gamma_{33}^1 = -r S \sin^2 \theta e^{-2\Psi} (r S_{,r} + S), \\ \Gamma_{02}^2 &= \frac{S_{,t}}{S}, \quad \Gamma_{12}^2 = \frac{r S_{,r} + S}{r S}, \quad \Gamma_{33}^2 = -\sin \theta \cos \theta, \\ \Gamma_{03}^3 &= \frac{S_{,t}}{S}, \quad \Gamma_{13}^3 = \frac{r S_{,r} + S}{r S}, \quad \Gamma_{23}^3 = \frac{\cos \theta}{\sin \theta}, \end{aligned} \quad (\text{A.2})$$

while the non-vanishing components of the Einstein tensor are given by

$$\begin{aligned} G_{tt} &= -\frac{e^{-2\Psi}}{r^2 S^2} \left\{ e^{2\Phi} \left[ 2r^2 S S_{,rr} + r S_{,r} (r S_{,r} + 6S) - 2r S (r S_{,r} + S) \Psi_{,r} + S^2 - e^{2\Psi} \right] \right. \\ &\quad \left. - r^2 e^{2\Psi} S_{,t} (2S \Psi_{,t} + S_{,t}) \right\}, \end{aligned} \quad (\text{A.3})$$

$$G_{tr} = -\frac{2}{r S} \left[ r S_{,tr} - (r S_{,r} + S) \Psi_{,t} - S_{,t} (r \Phi_{,r} - 1) \right], \quad (\text{A.4})$$

$$\begin{aligned} G_{rr} &= \frac{e^{-2\Phi}}{r^2 S^2} \left\{ e^{2\Phi} \left[ 2r S (r S_{,r} + S) \Phi_{,r} + r S_{,r} (r S_{,r} + 2S) + S^2 - e^{2\Psi} \right] \right. \\ &\quad \left. - r^2 e^{2\Psi} [2S S_{,tt} + S_{,t} (S_{,t} - 2S \Phi_{,t})] \right\}, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} G_{\theta\theta} &= r S e^{-2(\Phi+\Psi)} \left\{ e^{2\Phi} \left[ r (S \Phi_{,rr} + S_{,rr}) + r S \Phi_{,r} (\Phi_{,r} - \Psi_{,r}) \right. \right. \\ &\quad \left. \left. + (r S_{,r} + S) (\Phi_{,r} - \Psi_{,r}) + 2S_{,r} \right] \right. \\ &\quad \left. - r e^{2\Psi} [S \Psi_{,tt} + S_{,tt} - (S \Psi_{,t} + S_{,t}) (\Phi_{,t} - \Psi_{,t})] \right\}, \end{aligned} \quad (\text{A.6})$$

where  $(\ )_{,\alpha} \equiv \partial(\ )/\partial x^\alpha$ , ect.

### A. Solutions with Self-Similarity of the zeroth Kind

To study solutions with kinematic self-similarity of the zeroth kind, let us introduce two new dimensionless variables,  $x$  and  $\tau$ , via the relations

$$x = \ln(\xi) = -t + \ln(r), \quad \tau = t, \quad (\text{A.7})$$

or inversely

$$t = \tau, \quad r = e^{x+\tau}. \quad (\text{A.8})$$

Then, for any given function  $f(t, r)$  we find that

$$\begin{aligned} f_{,t} &= f_{,\tau} - f_{,x}, & f_{,r} &= \frac{1}{r} f_{,x}, \\ f_{,tr} &= -\frac{1}{r} (f_{,xx} - f_{,\tau x}), & f_{,rr} &= \frac{1}{r^2} (f_{,xx} - f_{,x}), \\ f_{,tt} &= f_{,\tau\tau} - 2f_{,\tau x} + f_{,xx}. \end{aligned} \quad (\text{A.9})$$

Substituting these expressions into Eqs.(A.3)-(A.6), we find that

$$\begin{aligned} G_{tt} &= -\frac{e^{-2\Psi}}{r^2 S^2} \left\{ e^{2\Phi} \left[ 2SS_{,xx} + S_{,x} (S_{,x} + 4S) - 2S\Psi_{,x} (S_{,x} + S) + S^2 - e^{2\Psi} \right] \right. \\ &\quad \left. - r^2 e^{2\Psi} (S_{,\tau} - S_{,x}) [2S (\Psi_{,\tau} - \Psi_{,x}) + (S_{,\tau} - S_{,x})] \right\}, \end{aligned} \quad (\text{A.10})$$

$$G_{tr} = \frac{2}{rS} [S_{,xx} - S_{,\tau x} + (S_{,x} + S) (\Psi_{,\tau} - \Psi_{,x}) + (S_{,\tau} - S_{,x}) (\Phi_{,x} - 1)], \quad (\text{A.11})$$

$$\begin{aligned} G_{rr} &= \frac{e^{-2\Phi}}{r^2 S^2} \left\{ e^{2\Phi} \left[ 2S\Phi_{,x} (S_{,x} + S) + S_{,x} (S_{,x} + 2S) + S^2 - e^{2\Psi} \right] \right. \\ &\quad \left. - r^2 e^{2\Psi} [2S (S_{,\tau\tau} - 2S_{,\tau x} + S_{,xx}) \right. \\ &\quad \left. + (S_{,\tau} - S_{,x}) (S_{,\tau} - 2S\Phi_{,\tau} - S_{,x} + 2S\Phi_{,x})] \right\}, \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned} G_{\theta\theta} &= S e^{-2(\Phi+\Psi)} \left\{ e^{2\Phi} [S\Phi_{,xx} + S_{,xx} + (\Phi_{,x} - \Psi_{,x}) (S\Phi_{,x} + S_{,x} + S) + S_{,x} - S\Phi_{,x}] \right. \\ &\quad \left. - r^2 e^{2\Psi} [S\Psi_{,xx} + S_{,xx} + S (\Psi_{,\tau\tau} - 2\Psi_{,\tau x}) + (S_{,\tau\tau} - 2S_{,\tau x}) \right. \\ &\quad \left. - (S\Psi_{,\tau} + S_{,\tau} - S\Psi_{,x} - S_{,x}) (\Phi_{,\tau} - \Psi_{,\tau} - \Phi_{,x} + \Psi_{,x})] \right\}. \end{aligned} \quad (\text{A.13})$$

For the solutions with self-similarity of the zeroth kind, the metric coefficients  $\Phi$ ,  $\Psi$  and  $S$  are functions of  $x$  only,

$$\Phi(\tau, x) = \Phi(x), \quad \Psi(\tau, x) = \Phi(x), \quad S(\tau, x) = S(x). \quad (\text{A.14})$$

Then, all the derivatives of these functions with respect to  $\tau$  are zero, and Eqs.(A.10) - (A.13) reduce to,

$$G_{tt} = \frac{e^{-2\Psi}}{r^2} \left\{ e^{2\Phi} \left[ 2y_{,x} - 2(1+y)\Psi_{,x} + 3y^2 + 4y + 1 - S^{-2}e^{2\Psi} \right] - r^2 e^{2\Psi} y (2\Psi_{,x} + y) \right\}, \quad (\text{A.15})$$

$$G_{tr} = \frac{2}{r} [y_{,x} - (1+y)(\Psi_{,x} - y) - y\Phi_{,x}], \quad (\text{A.16})$$

$$G_{rr} = \frac{e^{-2\Phi}}{r^2} \left\{ e^{2\Phi} \left[ (1+y)(2\Phi_{,x} + y + 1) - S^{-2}e^{2\Psi} \right] - r^2 e^{2\Psi} (2y_{,x} - 2y\Phi_{,x} + 3y^2) \right\}, \quad (\text{A.17})$$

$$G_{\theta\theta} = S^2 e^{-2(\Phi+\Psi)} \left\{ e^{2\Phi} [\Phi_{,xx} + y_{,x} + \Phi_{,x}(\Phi_{,x} - \Psi_{,x} + y) - (1+y)(\Psi_{,x} - y)] - r^2 e^{2\Psi} [\Psi_{,xx} + y_{,x} + (\Psi_{,x} - \Phi_{,x})(\Psi_{,x} + y) + y^2] \right\}, \quad (\text{A.18})$$

where

$$y = \frac{S_{,x}}{S}. \quad (\text{A.19})$$

For a perfect fluid, the energy-momentum tensor (EMT) takes the form,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \quad (\text{A.20})$$

where, when the fluid is co-moving with the frame of the coordinates, its four-velocity,  $u_\mu$ , is given by

$$u_\mu = r_1 e^\Phi \delta_\mu^t. \quad (\text{A.21})$$

Then, from the 01-component of the Einstein field equations  $G_{\mu\nu} = T_{\mu\nu}$ , and Eq.(A.16) we find that

$$y_{,x} - (1+y)(\Psi_{,x} - y) - y\Phi_{,x} = 0, \quad (G_{01} = T_{01}), \quad (\text{A.22})$$

while the other components yield

$$\rho = \frac{1}{(r_1 r)^2} \left\{ e^{-2\Psi} \left( 2y\Phi_{,x} + (1+y)^2 - S^{-2}e^{2\Psi} \right) - r^2 e^{-2\Phi} (2\Psi_{,x} + y)y \right\}, \quad (G_{00} = T_{00}), \quad (\text{A.23})$$

$$p = \frac{1}{(r_1 r)^2} \left\{ e^{-2\Psi} \left[ (1+y)(2\Phi_{,x} + y + 1) - S^{-2}e^{2\Psi} \right] - r^2 e^{-2\Phi} [2(1+y)\Psi_{,x} + y(y-2)] \right\}, \quad (G_{11} = T_{11}), \quad (\text{A.24})$$

$$p = \frac{1}{(r_1 r)^2} \left\{ e^{-2\Psi} [\Phi_{,xx} + \Phi_{,x}(\Phi_{,x} - \Psi_{,x} + 2y)] - r^2 e^{-2\Phi} [\Psi_{,xx} + \Psi_{,x}(\Psi_{,x} - \Phi_{,x} + 2y + 1) - y] \right\}, \quad (G_{22} = T_{22}). \quad (\text{A.25})$$

Note that in writing the above equations, Eq.(A.22) was used. To have Eqs.(A.24) and (A.25) be consistent, we must have

$$\Psi_{,xx} + \Psi_{,x}(\Psi_{,x} - \Phi_{,x} - 1) - y(y-1) = 0, \quad (\text{A.26})$$

$$\Phi_{,xx} + \Phi_{,x}(\Phi_{,x} - \Psi_{,x} - 2) - (1+y)^2 + S^{-2}e^{2\Psi} = 0, \quad (\alpha = 0). \quad (\text{A.27})$$

## B. Solutions with Self-Similarity of the Second Kind

To study solutions with kinematic self-similarity of the second kind, let us introduce other two new dimensionless variables,  $x$  and  $\tau$ , via the relations

$$x = \ln \left[ \frac{r}{(-t)^{1/\alpha}} \right], \quad \tau = -\ln(-t), \quad (\text{A.28})$$

or inversely

$$t = -e^{-\tau}, \quad r = e^{x-\tau/\alpha}, \quad (\text{A.29})$$

where  $\alpha$  is a *dimensionless* constant. Then, for any given function  $f(t, r)$  we find that

$$\begin{aligned} f_{,t} &= -\frac{1}{\alpha t} (\alpha f_{,\tau} + f_{,x}), & f_{,r} &= \frac{1}{r} f_{,x}, \\ f_{,tr} &= -\frac{1}{\alpha tr} (f_{,xx} + \alpha f_{,\tau x}), & f_{,rr} &= \frac{1}{r^2} (f_{,xx} - f_{,x}), \\ f_{,tt} &= \frac{1}{\alpha^2 t^2} (\alpha^2 f_{,\tau\tau} + 2\alpha f_{,\tau x} + f_{,xx} + \alpha^2 f_{,\tau} + \alpha f_{,x}). \end{aligned} \quad (\text{A.30})$$

Substituting these expressions into Eqs.(A.3)-(A.6), we find that

$$G_{tt} = -\frac{e^{-2\Psi}}{\alpha^2 r^2 S^2} \left\{ \alpha^2 e^{2\Phi} [2SS_{,xx} + S_{,x}(S_{,x} + 4S) - 2S\Psi_{,x}(S_{,x} + S) + S^2 - e^{2\Psi}] \right. \\ \left. - \frac{r^2}{t^2} e^{2\Psi} (2S\Psi_{,x} + S_{,x}) S_{,x} \right. \\ \left. - \frac{\alpha r^2}{t^2} e^{2\Psi} [2S(\alpha S_{,\tau}\Psi_{,\tau} + S_{,\tau}\Psi_{,x} + \Psi_{,\tau}S_{,x}) + S_{,\tau}(\alpha S_{,\tau} + 2S_{,x})] \right\}, \quad (\text{A.31})$$

$$G_{tr} = \frac{2}{\alpha r t S} \{ S_{,xx} - \Psi_{,x}(S_{,x} + S) - S_{,x}(\Phi_{,x} - 1) \\ + \alpha [S_{,\tau x} - \Psi_{,\tau}(S_{,x} + S) - S_{,\tau}(\Phi_{,x} - 1)] \}, \quad (\text{A.32})$$

$$G_{rr} = \frac{e^{-2\Phi}}{\alpha^2 r^2 S^2} \left\{ \alpha^2 e^{2\Phi} [2S\Phi_{,x}(S_{,x} + S) + S_{,x}(S_{,x} + 2S) + S^2 - e^{2\Psi}] \right. \\ \left. - \frac{r^2}{t^2} e^{2\Psi} [2SS_{,xx} + S_{,x}(S_{,x} - 2S\Phi_{,x} + 2\alpha S)] \right. \\ \left. - \frac{\alpha r^2}{t^2} e^{2\Psi} [2S(\alpha S_{,\tau\tau} + 2S_{,\tau x}) + S_{,x}(S_{,\tau} - 2S\Phi_{,\tau}) \right. \\ \left. + S_{,\tau}(\alpha S_{,\tau} - 2\alpha S\Phi_{,\tau} + S_{,x} - 2S\Phi_{,x} + 2\alpha S)] \right\}, \quad (\text{A.33})$$

$$G_{\theta\theta} = \frac{S}{\alpha^2} \left\{ \alpha^2 e^{-2\Psi} [S\Phi_{,xx} + S_{,xx} + (\Phi_{,x} - \Psi_{,x})(S\Phi_{,x} + S_{,x} + S) - S\Phi_{,x} + S_{,x}] \right. \\ \left. - \frac{r^2}{t^2} e^{-2\Phi} [S\Psi_{,xx} + S_{,xx} - (S\Psi_{,x} + S_{,x})(\Phi_{,x} - \Psi_{,x} - \alpha)] \right. \\ \left. - \frac{\alpha r^2}{t^2} e^{-2\Phi} [S(\alpha\Psi_{,\tau\tau} + 2\Psi_{,\tau x}) + \alpha S_{,\tau\tau} + 2S_{,\tau x} \right. \\ \left. - (S\Psi_{,\tau} + S_{,\tau})(\alpha\Phi_{,\tau} - \alpha\Psi_{,\tau} + \Phi_{,x} - \Psi_{,x} - \alpha) \right. \\ \left. - (\Phi_{,\tau} - \Psi_{,\tau})(S\Psi_{,x} + S_{,x}) \right\}. \quad (\text{A.34})$$

For the solutions with self-similarity of the second kind, the metric coefficients are also functions of  $x$  only, but now with  $x$  being given by Eq.(A.28). Then, setting all the derivatives with respect to  $\tau$  zero, Eqs.(A.31) - (A.34) reduce to

$$G_{tt} = -\frac{1}{r^2} e^{2(\Phi-\Psi)} [2y_{,x} + y(3y + 4) + 1 - 2(1 + y)\Psi_{,x} - S^{-2}e^{2\Psi}] \\ + \frac{1}{\alpha^2 t^2} (2\Psi_{,x} + y)y, \quad (\text{A.35})$$

$$G_{tr} = \frac{2}{\alpha t r} [y_{,x} + (1 + y)(y - \Psi_{,x}) - y\Phi_{,x}], \quad (\text{A.36})$$

$$G_{rr} = \frac{1}{r^2} [2(1 + y)\Phi_{,x} + (1 + y)^2 - S^{-2}e^{2\Psi}] \\ - \frac{1}{\alpha^2 t^2} e^{2(\Psi-\Phi)} [2y_{,x} + y(3y - 2\Phi_{,x} + 2\alpha)], \quad (\text{A.37})$$

$$G_{\theta\theta} = S^2 e^{-2\Psi} [\Phi_{,xx} + y_{,x} + \Phi_{,x} (\Phi_{,x} - \Psi_{,x} + y) + (1+y)(y - \Psi_{,x})] - \frac{r^2 S^2}{\alpha^2 t^2} e^{-2\Phi} [\Psi_{,xx} + y_{,x} + y^2 - (\Psi_{,x} + y)(\Phi_{,x} - \Psi_{,x} - \alpha)]. \quad (\text{A.38})$$

For a perfect fluid, the EMT is given by Eqs.(A.20) and (A.21). Similarly, from the 01-component of the Einstein field equations we find that

$$y_{,x} - (1+y)(\Psi_{,x} - y) - y\Phi_{,x} = 0, \quad (G_{01} = T_{01}), \quad (\text{A.39})$$

while the other components yield

$$\rho = \frac{1}{\kappa r_1^2} \left\{ \frac{1}{\alpha^2 t^2} e^{-2\Phi} (2\Psi_{,x} + y) y - \frac{1}{r^2} e^{-2\Psi} [2y\Phi_{,x} + (1+y)^2 - S^{-2} e^{2\Psi}] \right\}, \quad (G_{00} = T_{00}), \quad (\text{A.40})$$

$$p = \frac{1}{\kappa r_1^2} \left\{ \frac{1}{r^2} e^{-2\Psi} [2(1+y)\Phi_{,x} + (1+y)^2 - S^{-2} e^{2\Psi}] - \frac{1}{\alpha^2 t^2} e^{-2\Phi} [2(1+y)\Psi_{,x} + y^2 + 2(\alpha-1)y] \right\}, \quad (G_{11} = T_{11}), \quad (\text{A.41})$$

$$p = \frac{1}{\kappa r_1^2} \left\{ \frac{1}{r^2} e^{-2\Psi} [\Phi_{,xx} + \Phi_{,x} (\Phi_{,x} - \Psi_{,x} + 2y)] - \frac{1}{\alpha^2 t^2} e^{-2\Phi} [\Psi_{,xx} + \Psi_{,x} (\Psi_{,x} - \Phi_{,x} + 2y + \alpha + 1) + (\alpha-1)y] \right\}, \quad (G_{22} = T_{22}), \quad (\text{A.42})$$

where in writing the above equations, Eq.(A.39) was used.

When  $\alpha \neq 1$ , in the expressions of  $p$  the term that is proportional to  $r^{-2}$  has different power-dependence on  $r$  from the term that is proportional to  $t^{-2}$ , when these expressions are written in terms of  $r$  and  $x$ , since

$$t = -r^\alpha e^{-\alpha x}. \quad (\text{A.43})$$

Then, the two expressions of Eqs.(A.41) and (A.42) are equal only when

$$\Psi_{,xx} + \Psi_{,x} (\Psi_{,x} - \Phi_{,x}) + (\alpha-1)(\Psi_{,x} - y) - y^2 = 0, \quad (\text{A.44})$$

$$\Phi_{,xx} + \Phi_{,x} (\Phi_{,x} - \Psi_{,x} - 2) - (1+y)^2 + S^{-2} e^{2\Psi} = 0. \quad (\text{A.45})$$

When  $\alpha = 1$ , all the terms in the expressions of  $p$  have the same power-dependence on  $r$ , and the two expressions of Eqs.(A.41) and (A.42) are equal, provided that

$$\begin{aligned} & \Phi_{,xx} + \Phi_{,x} (\Phi_{,x} - \Psi_{,x} - 2) - (1 + y)^2 + S^{-2} e^{2\Psi} \\ & - e^{2(x+\Psi-\Phi)} \left[ \Psi_{,xx} + \Psi_{,x} (\Psi_{,x} - \Phi_{,x}) - y^2 \right] = 0, \quad (\alpha = 1). \end{aligned} \quad (\text{A.46})$$

From the above equations we can see that a solution that satisfies Eqs.(A.44) and (A.45) with  $\alpha = 1$  is also a solution of Eq.(A.46), but not the other way around, that is, a solution of Eq.(A.46) doesn't necessarily satisfy Eqs.(A.44) and (A.45).



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